ABSTRACT
This paper presents a numerical modeling of the collision between a small bubble –of a few hundred microns, initially moving at terminal velocity, and an inclined wall, with relevance to drag reduction schemes. The theoretical model uses the lubrication theory to describe the film drainage as the bubble approaches the wall, and compute the force exerted by the wall as the integral of the excess pressure due to the bubble deformation. The model is solved using finite differences. The trajectory of the bubble is then determined using equations of classical mechanics. This study is an extension of previous work by Moraga, Cancelos and Lahey, [Multiphase Science and Technology, 18 ,(2),2006] where the simulation and comparison with experiments was carried out for a horizontal wall. In the present study where the wall is inclined, the bubble trajectory is no longer one-dimensional and axisymmetry around the vertical axis is lost, allowing for more complex behavior. The influence of various parameters (Reynolds number, Weber number) is examined. Numerical results are compared with the experimental data from Tsao and Koch [Physics Fluids 9, 44, 1997]

1-INTRODUCTION
Drag reduction by changing the nature of the fluid and the turbulence associated with it is an idea that has received its share of attention over the years. Back in 1985, Lumley and Kubo [1] observed that injecting polymers seemed to break down the generation of turbulence in the wall layer. More recently, experiments have shown evidence of some drag reduction –typically a few % - on full-scale flat plate turbulence [2] such as would be found along a ship’s hull. Injecting micro-bubbles at the base of a Taylor-Couette cell has led to drag reduction of up to 20%[3]. Yet understanding the nature of the
interaction between the bubbles and the wall remains a challenge, particularly as it involves a wide variety of scales. Antal et al. [4] have derived a popular model for two-phase flows, however theirs is only valid for vertical walls. Moraga et al [5] have developed a model for inclined walls and deformable bubbles, but the model fails to account explicitly for the parameters of the flow such as the Reynolds number or the Weber number. Following an approach similar to Klaseboer et al. [6], Moraga et al. [7] have used lubrication theory to build an approximation for the force exerted by the wall on the bubble over a range of Reynolds numbers. The idea is to represent the effect of the wall via a pressure force corresponding to the drainage of the film between the bubble and the wall. The equations of motion in the viscous film describe the evolution of the film height (i.e., the interface fluid/bubble) and are coupled with the trajectory of the bubble centroid through the pressure-dependent wall force and the kinematic condition on the boundary of the interface. By integration of the equations the time evolution of the wall force can be determined. Moraga et al [7] then derived a simple, parameter-based model for the wall force which could in turn be integrated into more complex simulations for bubbly flows. In its present formulation, the model has only been derived in the axisymmetrical case (one dimension). This is not adequate to describe inclined walls – such as those along a ship’s hull – as axisymmetry no longer holds. The purpose of the paper is to evaluate to which extent a two dimensional extension of this approach can reproduce the dynamics of a bubble interacting with an inclined wall. This validation is a first step toward better modeling of bubbly flows near walls.

The paper is organized as follows: we describe the equations in section 2. Practical and theoretical aspects of the implementation are detailed in section 3. Results for a horizontal wall and an inclined wall are presented in sections 4 and 5. A conclusion is given in section 6.

2-THE EQUATIONS OF MOTION

The problem is to describe the motion of a bubble of density \( \rho_B \), of volume \( V \) and equivalent radius \( R \), rising initially at terminal velocity \( V_t \) in a quiescent fluid towards a wall inclined at angle \( \theta \) (see figure 1).

Let \( \rho_l \) and \( \mu_l \) be respectively the fluid density and viscosity. One can define a Reynolds number associated with the flow

\[
\text{Re} = \frac{2 \rho_l V_t R}{\mu_l}
\]

As the bubble approaches the wall, surface tension plays an important role since the moving bubble interacts with the wall and deforms. Let us define the Weber number as

\[
We = \frac{\rho_l V_t^2 R}{\sigma}
\]
We choose to describe the motion in a referential moving with the bubble tangential velocity relative to the wall. We assume that the bubble tangential acceleration is small enough to be neglected, so that the referential can be assumed to be Galilean.

Let \((i,j,k)\) – resp. \((x,y,z)\) – be the coordinate system – resp. coordinate variables – associated with the referential. The origin \(O\) of the referential coincides with the normal projection of the bubble centroid onto the wall.

Let us write the bubble centroid velocity \(\mathbf{U}\) as \(\mathbf{U} = U_i + V_j\)

The equations of motion for the bubble centroid are obtained by estimating the different forces acting on the bubble:

\[
V \frac{dU}{dt} = F_{\text{buoyancy}} + F_{\text{drag}} + F_{\text{AM}} + F_{\text{wall}}
\]

where \(F_{\text{buoyancy}}\), \(F_{\text{drag}}\) and \(F_{\text{AM}}\) are respectively the buoyancy, drag and added mass forces, with

\[
F_{\text{buoyancy}} = (\rho_l - \rho_b) \frac{4}{3} \pi R^3 g \sin \theta_i
\]

\[
- (\rho_l - \rho_b) \frac{4}{3} \pi R^3 g \cos \theta_j
\]

\[
F_{\text{drag}} = -C_d \text{Re} 4 \pi R U_i
\]

\[
- C_d \text{Re} 4 \pi R V_j
\]

\[
F_{\text{added mass}} = -C_{\text{vm}} \rho_l \left( \frac{dU}{dt} i + \frac{dV}{dt} j \right)
\]

In the above equations, \(g\) is the gravity, \(C_d\) is the drag coefficient for the bubble in the absence of the wall, \(C_{\text{vm}}\) is the bubble added mass coefficient.

Note that following Moraga et al. [7], we have not included the history force in the equations, as it is difficult to model accurately and Klaseboer et al. [6] have found that it did not seem to have a strong influence on the bubble dynamics.

The effect of the wall is represented by a force \(F_{\text{wall}}\). The idea is that the wall makes itself felt through an excess pressure exerted on the top of the bubble, which corresponds to a deformation of the interface. The flow between the bubble and the wall constitutes a film which can be described using lubrication theory. The deformation corresponds to the height \(h(x,y)\) of the film between the bubble and the wall, since the effects of deformation at the interface farther from the wall are neglected by the model.

Let \(P_B\) be the pressure inside the bubble and \(P_L\) be the pressure just outside the bubble interface. Initially, in the case of a spherical bubble, one has

\[
P_L - P_B = - \sigma \left( \frac{2}{R} \right)
\]

As the bubble deforms, the pressure jump becomes

\[
P_L - P_B = - \sigma \left( \frac{1}{R_x} + \frac{1}{R_z} \right)
\]

where \(R_x\) and \(R_y\) are the principal curvature radii of the deformed interface. The effect of the wall can therefore be computed as the spatial integral of the pressure difference between the spherical and the deformed interface. The deformation is assumed to take place only on the top surface of the bubble within a region of size \(|x|, |z| < r_{\text{max}}\) with \(r_{\text{max}} \ll R\). In practice, we take \(r_{\text{max}} \approx R\) and we check \textit{a posteriori} that the excess pressure is indeed negligible on most of this domain. This excess pressure \(\Delta p\) can be written as

\[
\Delta p = \sigma \left( -\left( \frac{1}{R_x} + \frac{1}{R_z} \right) + \frac{2}{R} \right)
\]
For a slowly deforming interface or film height of the form \( h(x,z) \) one has approximately
\[
\frac{1}{R_x} + \frac{1}{R_z} = \frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial z^2} \quad (6)
\]

Finally, the pressure force can be expressed as
\[
F_{\text{wall}} = \left( \int (\Delta p) n_x dx dz \right) \hat{i} + \left( \int (\Delta p) n_z dx dz \right) \hat{j} \quad (7)
\]

where \( n_x \) and \( n_z \) are the components of the bubble normal, and the normal can be approximated by
\[
n = \frac{\partial h}{\partial x} \hat{i} + \frac{\partial h}{\partial z} \hat{k} - \hat{j}
\]

Combining equations (2) to (7) together, one can then rewrite (1) as
\[
\frac{4}{3} \pi R^3 (C_{vis} + \frac{\rho_b}{\rho_l}) \frac{dU}{dt} = (\rho_l - \rho_b) \frac{4}{3} \pi R^3 g \sin \theta
\]
\[
-C_d \Re 4 \pi RU + \sigma \int \left( \frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial z^2} \right) + \frac{2}{R} \frac{\partial h}{\partial x} dx dz
\]

(10)

\[
\frac{4}{3} \pi R^3 (C_{vis} + \frac{\rho_b}{\rho_l}) \frac{dV}{dt} = -(\rho_l - \rho_b) \frac{4}{3} \pi R^3 g \cos \theta
\]
\[
-C_d \Re 4 \pi RV - \sigma \int \left( \frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial z^2} \right) - \frac{2}{R} \frac{\partial h}{\partial x} dx dz
\]

(11)

In the referential, at \( t=0 \)
\[
U = V \sin \theta \quad (12)
\]
\[
V = -V \cos \theta \quad (13)
\]

To solve for the bubble centroid velocity in (10)-(11) with initial conditions (12) and (13), we need an evolution equation for the liquid film height as it is drained between the bubble and the wall. This is readily obtained from lubrication theory, in which we assume that the flow in the film is in the plane of the wall and dominated by viscous effects. Given that even a very low concentration of impurities in the flow is sufficient to make the the interface immobile, we assume a no-slip boundary condition for the liquid at the bubble interface, rather than a no-shear condition (see for instance Klaseboer et al. [6], or Lin et al. [8] for a full discussion).

Mass conservation for a slab of fluid within the film yields the modified lubrication equation (see also Moraga et al. [6])
\[
\frac{\partial h}{\partial t} + \frac{1}{2} \frac{\partial}{\partial x} (Uh) = \frac{1}{12 \mu} \frac{\partial}{\partial x} (h^3 \frac{\partial P}{\partial x}) + \frac{1}{12 \mu} \frac{\partial}{\partial y} (h^3 \frac{\partial P}{\partial y}) \quad (14)
\]

where \( P_L \) is the pressure in the liquid film. Assuming that the pressure variations within the bubble are negligible compared to those in the film, one can replace the film pressure in (14) with the pressure difference
\[
P_t - P_B = -\sigma ((\frac{1}{R_x}) + (\frac{1}{R_y})) \quad (15)
\]

Substituting (15) in (14) leads to a fourth-order elliptic equation which requires two boundary conditions on the boundary at all times. These are provided by the fact that outside the film domain the pressure difference is zero and the bubble is no longer deformed by the wall i.e it moves with the centroid velocity. This means that on the boundary \( (x^2 + z^2)^{1/2} = r_{max} \)
\[
\Delta p = 0 \quad (16)
\]
\[
\frac{dh}{dt} = -V \quad (17)
\]

Note that if we assume that the interface is mobile, we obtain a modified equation for the film height
\[
\frac{\partial h}{\partial t} + \frac{1}{2} \frac{\partial}{\partial x} (Uh) = \frac{1}{3 \mu} \frac{\partial}{\partial x} (h^3 \frac{\partial P}{\partial x}) + \frac{1}{3 \mu} \frac{\partial}{\partial y} (h^3 \frac{\partial P}{\partial y}) \quad (14')
\]

Since the bubble is initially spherical, the initial condition for the film is
\[ h(x, z) = h_0 + R - (R^2 - (x^2 + z^2))^{0.5} \]

\[ h(x, z) = h_0 - \frac{x^2 + z^2}{2R} \quad (18) \]

and we have
\[ \Delta p = 0 \quad (19) \]

Equations (10)(11)(14)(15) together with boundary conditions (12)(13)(16)(17) (18)(19) constitute a coupled system (S) in which the bubble centroid position \((U,V)\) and its top interface \(h(x, z)\) are determined. Numerical solving of the system (S) is discussed in the next section.

3-NUMERICAL IMPLEMENTATION

We emphasize the specific features of our simulations. The reader is referred to Moraga’s paper [7] for a general presentation. The main difference of this work with that of Moraga et al [7] is that we solve a two dimensional version of the lubrication and bubble trajectory equation as needed due to the fact that the inclined wall breaks the axial symmetry of the horizontal wall case. The equations constituting (S) are discretized using second-order finite differences and linearized about the current solution. The system is no longer pentadiagonal as in 1-D, but has thirteen non-zero diagonals over a bandwith of \(2N+1\), where \(N\) is the number of grid points in either direction \(x\) or \(z\). NAG Fortran routine were used to solve for the band-diagonal system within a tolerance of \(10^{-8}\). The code was tested for different spatial resolutions. We used the same time step as Moraga et al. [7]. An increase of 30\% in the spatial resolution did not produce any noticeable changes. The impact of larger increases in the spatial resolution were not tested due to the large increase in computational time associated with improved resolution.

The lubrication equations are discretized in cartesian coordinates, in which defining a circular or elliptic boundary was not straightforward. We chose instead to define a square boundary interface
\[ |x|=r_{max}, |z|=r_{max} \]

We expect the pressure and interface variations to be significant only near the very top part of the bubble, so that the exact contour of the interface boundary should be irrelevant. This hypothesis was checked \textit{a posteriori}.

A variety of models were used to represent the drag and the added mass coefficients, as well as the correction due to the influence of the wall.

For spherical quasi-rigid bubbles, we used Schiller and Nauman’s law [9]
\[ C_D \ Re = 24(1 + 0.15 \ Re^{0.687}) \]

which has been validated for Reynolds numbers up to 1000.

In order to account for the effect of the wall on the drag coefficient, following [7], this law was replaced by
\[ C_D \ Re = 24C(y)(1 + 0.15 \ Re^{0.687}) \]

where
\[ C(y) = \frac{1}{(1 - \frac{R}{2y})^3} \]

For a spherical rigid bubble, the added mass coefficient is equal to 0.5.

If the effect of the wall is taken into account, one then has [10]
\[ C_{VM} = \frac{3}{2} \frac{1}{\left(1 - \frac{R}{2y}\right)^3} - 1 \]

For non-spherical bubbles of aspect ratio \(\zeta\), we used Moore’s functions [11]
\[ C_D \ Re = 55G(\zeta)(1 + H(\zeta)Re^{-0.5}) \]

where
\[ G(\zeta) = \frac{3}{3} (\zeta^2 - 1) \frac{\sqrt{\zeta^2 - 1} - (-\zeta^2 + 2) \cos^{-1} \zeta^{-1}}{(-\sqrt{\zeta^2 - 1} + \zeta^2 \cos^{-1} \zeta^{-1})^2} \]

and
$H(\zeta) = 0.0019\zeta^4 - 0.21\zeta^3 + 1.70\zeta^2 - 2.15\zeta - 1.57$

For a bubble of aspect ratio $\zeta < 2.5$, the added mass coefficient is (see [6])

$C_m = 0.62\zeta - 0.12$

The aspect ratio of the bubbles was kept constant throughout the simulation, identical to the initial aspect ratio observed in similar cases experimentally. However, this is different from what usually happens in experiments where the aspect ratio of the bubble changes sharply after the rebounds (see e.g. [12]). The reason to keep the aspect ratio constant is that the correlations above account for the effect of drag for a bubble away from walls, where the aspect ratio is due to the competition of surface tension and buoyancy. But when a wall is present, the wall also contributes to deform the bubble in ways not accounted for in the expansions derived by Moore.

For a particular bubble in a given fluid – here we focus in air bubbles rising in water – knowing the bubble radius should be sufficient to determine the terminal velocity from balancing the drag force with the buoyancy force. However, since the terminal velocity is sensitive to the presence of impurities ([13], [9]), we restricted our study to experimentally determined terminal velocities such as the experiments of Tsao and Koch [14], which describe the motion of air bubbles rising in water, or the experimental visualizations carried out by our group [12]. In the following sections, we selected two cases from the Tsao and Koch database and compared it with results predicted by our simulation.

**4-RESULTS FOR A FLAT WALL**

A number of tests were performed to validate the code for a horizontal wall. We choose to show results for one case involving large bubbles documented by Tsao and Koch [14] – which we will denote TK1. The bubble radius is 0.79 mm and the Reynolds number and Weber number are respectively 420 and 0.74 - note that our Weber number is defined as half that of Tsao and Koch.

Moraga et al. have compared the 1-D code with TK1, and found evidence of agreement for trajectories and time scales. Since in this case the wall is horizontal, this is an opportunity to compare the 2-D code with its 1-D version. Figure 2 show that in this case both codes give similar results.

![Figure 2](image_url)

**Fig. 2.** Vertical position of the bubble centroid – the wall is at y=0mm

The flow naturally remains axisymmetric. However, the use of square boundary conditions in the 2-D code could have generated some discrepancy, which was not the case, as can be seen in figure 3, which shows the adimensional excess pressure defined as

$$\Delta p^* = \Delta p \frac{R}{5\sigma}$$
The pressure variations are concentrated in a region close to the top of the bubble so that boundary conditions at the exact boundary are not significant.

**Fig. 3:** Representation of the adimensional excess pressure $\Delta p^*$; Solid line: 1-D code with 200 radial nodes; Dashed line: 2-D code with 80x80 nodes

Furthermore, the much coarser resolution (by a factor of 5) in the 2-D case does not appear to perturb significantly the pressure profiles. Owing to the complexity of the problem in 2-D, we have not yet attempted a resolution comparable to the 1-D case. As Moraga et al. [7] have already shown, figure 4 confirms that the force balance essentially sets between the added mass force and the wall force.

**Fig. 4:** Time evolution of the forces acting on the bubble – buoyancy has been omitted.

### 5-RESULTS FOR AN INCLINED WALL

We now present results for the 2-D code in another case documented by Tsao and Koch – which we will refer to as TK2. In the TK2 case a bubble of radius 0.65 mm rises towards a wall inclined at 60 degrees to the horizontal. The Reynolds number and Weber number are respectively 250 and 0.30 in our notation. Bouncing was observed in the experiment, which was indeed reproduced by our code (see figure 5). The characteristic time scale for the rebound is about 20 ms, which is on the order of that observed in Tsao and Koch’s experiment. Figure 5 shows a characteristic amplitude for the vertical velocity during the first rebound of about a few cm/s, in agreement with the value of 3 cm/s reported by Tsao and Koch.

**Fig. 5:** Normal velocity in cm/s of the bubble centroid in the TK2 case
Figure 6 shows that the tangential velocity exhibits small oscillations around its mean value. The mean value, amplitude and time frequency of the oscillations are quite similar to those measured by Tsao and Koch (see figure 9 in their paper). Figure 7a and 7b represent the force balance in the tangential and the normal directions x and y. In the tangential direction, the presence of the wall affects the drag and the added mass force only to a small extent, as could be expected from figure 6, since the amplitude of the velocity variations in the tangential direction are much less than in the normal direction.

Figure 8 represents the adimensional excess pressure $\Delta p^a$ along the bubble surface during the first rebound, at select times corresponding to the vertical bars plotted on figures 7b) and 5. The flow begins to lose its axisymmetry as the bubble approaches the wall. As the bubble moves away from the wall, the pressure distribution becomes more intense and strongly asymmetrical. Note that even as the pressure force decreases, the maximum intensities (both positive and negative) of the instantaneous pressure keep increasing. Once again the pressure only varies significantly on a fraction of the upper surface of the
bubble, which seems to justify a posteriori the approximation we made when using square boundary conditions. It would however be interesting to carry out the derivation in cylindrical coordinates in the near future using exact boundary conditions.

**Fig. 8**: Adimensional excess pressure in the TK2 case along the bubble surface
a) $t=21$ ms; b) $t=22.4$ ms; c) $t=23.8$ ms; d) $t=25.2$ ms; e) $t=26.8$ ms; the rebound takes place at $t=24.1$ ms.
6-CONCLUSION
2-D lubrication theory has been used in conjunction with a trajectory equation to describe the motion of a bubble as it approaches an inclined wall under the effect of buoyancy. The method had already been implemented in the axisymmetrical case [6, 7], which cannot be used for inclined walls. We have checked that in axisymmetrical cases the code gave similar results to its 1-D counterpart. Agreement with experimental data has been established in the cases of a horizontal as well as a steeply inclined wall.

REFERENCES


interacting with a rigid wall” Physics of Fluids 1997, 9, 44